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BIHAR

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FROM THE PEN OF THE GENERAL SECRETARY

There are various Mathematical organisations in India but there was none such in Bihar.

Teachers and students of Mathematics of Bihar needed an organisation to bring them together to discuss the achievements and problems of the different spheres of Mathematics. It was with this aim and object that the teachers of different local colleges of Bhagalpur assembled in the T. N. B. College premises on 1-2-58 under the presidentship of Dr. B. N. Prasad, D. Sc (Paris), Ph. D. (Liverpool), M. Sc., F. N. I and resolved to organise a society named "Bihar Mathematical Society".

Dr. R. Shukla, Ph D. (London), Head of the Deptt. of Math., L. S. College, Muzaffarpur, was elected President and Prof. Shree Nandan Prasad of T. N. B. College was elected the General Secretary.

Prof. K. D. Khemka was elected Treasurer of the Society. Later on Prof. Jai Narain, M. A. (London) Head of the Deptt. of Mathematics, T. N. B. College was made Vice-President.

The following *tentative constitution* of the society was framed.

1. The aims and objects of the Society shall be
   (a) to start a publication containing
      (i) Original research papers; (ii) Articles on Mathematical topics; and (iii) News of Mathematical interest; and
   (b) to organise from time to time lectures from distinguished mathematicians; and
   (c) to organise mathematical exhibitions.
2. The following type of memberships be introduced:
   (a) **Ordinary members** (Admission fee Rs. 5/- and annual subscription Rs. 6/-)

a.

(b) **Ordinary members for 5 yrs** (Subscription Rs. 27/-)

(c) **Ordinary members for 10 yrs.** (Subscription Rs. 51/-)

(d) **Donor members** (minimum contribution Rs. 50/-)

(e) **Life members** (minimum subscription Rs. 101/-)

(f) **Sessional members** (Subscription Rs. 12/- per session)

(g) **Founder members** (Minimum contribution Rs. 10/- over and above any type of above memberships)

(h **Student members** (Minimum contribution Rs. 2/-) per year

3. A member of the Society shall have the rights and privileges

(a) To contribute papers for reading at the Annual Session of the Society ;

(b) To receive free of charge any publication of the Society for the Session ;

Provided that the student member will be given copy or copies of the published journals at concession rates ;

(c) To seek election to any vacant office of the executive for the session [Except the donor members vide 2 (d)].

Teachers of Mathematics of all the colleges of Bihar were approached for becoming members. Some of them were enrolled but most of them were not enrolled. Some student members were also enrolled. In publishing this first issue of the Journal of the society we have received co-operation from all corners. Eminent mathematicians from outside and inside India have encouraged us by sending their valuable papers for publication in this issue of the Journal.

The proofs of the papers were read and corrected by the authors concerned and no paper has been examined by any Editorial Board this time.

We are highly obliged to the authorities of the Bhagalpur University for granting a suitable amount for the publication of this journal.

We would request the principals of all colleges of Bihar to purchase each a copy of the Journal for their college library. This will encourage us and at the same time the society would get monetary assistance for its proper functioning.

Papers and other contributions for the second issue of the journal may be sent to the general secretary.

Any suggestion for improvement of the journal will be appreciated.

TWO THEOREMS ON THE ORDER OF INTEGRALS

By B. N. PRASAD

Professor and Head, Mathematics Department, University of Allahabad

1. The theorems, which we want to prove, are the following :—

Theorem 1. If $f(x)$ be integrable (L) *in every finite interval, then as $x \to \infty$,*

$$F(x) = \int_A^x f(t)\, dt = o\,(x^\delta),$$

where $\delta > 0$, provided the integral

$$\int_A^\infty \frac{f(t)}{t^\delta}\, dt$$

exists as a finite number.

Theorem 2. If $f(x)$ be integrable (L) *in* $(0, x)$, then as $x \to 0$,

$$F(x) = \int_0^x f(t)\, dt = o\,(x^\delta),$$

where $\delta > 0$, provided

$$\lim_{\varepsilon \to 0} \int_\varepsilon^x \frac{f(t)}{t^\delta}\, dt$$

exists as a finite number.

These theorems do not seem to have been given previously in the general form as given here, though no doubt special cases* of them have been proved and used by various writers. The particular case of the Theorem 2 when $\delta = 1$, was given by Thomae†, on which a paper was published also by G. Prasad‡ in which he considered in

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*The only case of Theorem 1 that I have seen is that, as $x \to \infty$,

$$\int_A^x |f(t)|\, dt = o\,(x), \text{ provided the integral } \int_A^\infty \left| \frac{f(t)}{t} \right| dt$$

exists. See Hobson (1), 723.

†Thomae (3)

‡ G. Prasad (2).

great details a class of function $f(x)$ for which $F(x)$ may be **of the** type $o(x)$, even when

$$\lim_{\varepsilon \to 0} \int_{\varepsilon}^{x} \frac{f(t)}{t} dt$$

does not necessarily exist.

The case of the Theorem 2 when $\delta = 1$, may be looked upon still from another angle of view, namely, the *fundamental theorem of the Integral Calculus.* It gives a *sufficient* condition for the differentiability of the Integral of a function at a point of discontinuity of the second kind, on which topic there seems to be still scope for further work.

2. Proof of Theorem I. Let

$$X(x) = \int_{A}^{x} \frac{f(t)}{t^{\delta}} dt.$$

We have

$$\frac{1}{x^{\delta}} \int_{A}^{x} f(t)\, dt = \frac{1}{x^{\delta}} \int_{A}^{x} t^{\delta} \frac{f(t)}{t^{\delta}}\, dt$$

$$= X(x) - \frac{1}{x^{\delta}} \int_{A}^{x} \delta\, t^{\delta - 1}\, X(t)\, dt,$$

by integration by parts. Now corresponding to an arbitrarily small positive number ε, a number A' can be found such that for all values of $x \geqslant A'$, we can put

$$X(x) = X(A') + \eta,$$

where η may be given a suitable value such that $|\eta| < \varepsilon$. Now

$$\frac{1}{x^{\delta}} \int_{A}^{x} f(t)\, dt = X(x) - \frac{1}{x^{\delta}} \int_{A}^{A'} \delta\, t^{\delta - 1}\, X(t)\, dt - \frac{1}{x^{\delta}} \int_{A'}^{x} \delta X(t)\, t^{\delta - 1} dt.$$

Here

$$\frac{1}{x^{\delta}} \int_{A}^{A'} \delta t^{\delta - 1} X(t) dt \leqslant \frac{M}{x^{\delta}} \int_{A}^{A'} \delta t^{\delta - 1} dt = \frac{M}{x^{\delta}} (A'^{\delta} - A^{\delta}) = o(1),$$

as $x \to \infty$, M being the upper bound of $|X(t)|$ in (A, A').

Again, applying the first mean value theorem, we get

$$\frac{1}{x^{\delta}}\int_{A}^{x}\delta t^{\delta-1}\,X(t)dt = X(A'+\theta\{x-A'\}).\ \frac{1}{x^{\delta}}\int_{A'}^{x}\delta t^{\delta-1}\,dt = 0\leqslant\theta\leqslant 1,$$

$$= \{X(A')+\eta\}.\quad \frac{1}{x^{\delta}}\,(x^{\delta}-A'^{\delta})$$

$$= X\,(A') - \frac{X(A')\;A'^{\delta}}{x^{\delta}} + \eta\,\left(1-\frac{A'^{\delta}}{x^{\delta}}\right).$$

Hence, we have

$$\lim_{x\to\infty}\frac{1}{x^{\delta}}\int_{A}^{x} f(t)\,dt = \lim_{x\to\infty}\left[\{X(x)-X(A')\} + \frac{X(A')A'^{\delta}}{x^{\delta}} - \eta\left(1-\frac{A'^{\delta}}{x^{\delta}}\right)\right]$$

$$= 0.$$

This proves the theorem.

3. Proof of Theorem 2. Let

$$\phi\,(x) = \int_{\varepsilon}^{x}\frac{f(t)}{t^{\delta}}\,dt.$$

Then we have

$$\frac{1}{x^{\delta}}\int_{\varepsilon}^{x} f(t)\,dt = \frac{1}{x^{\delta}}\int_{\varepsilon}^{\delta} t^{\delta}\,\frac{f\,(t)}{t^{\delta}}\,dt.$$

$$= \phi(x) - \frac{1}{x^{\delta}}\int_{\varepsilon}^{\delta}\delta t^{\delta-1}\phi(t)dt$$

by integration by parts. Now using the first mean value theorem, we have

$$\frac{1}{x^{\delta}}\int_{\varepsilon}^{x} f(t)dt = \phi(x) - \phi(\xi)\ \frac{1}{x^{\delta}}\int_{\varepsilon}^{x}\delta t^{\delta-1}dt$$

$$= \phi(x) - \phi(\xi) + \phi(\xi).\quad \frac{\varepsilon^{\delta}}{x^{\delta}},$$

where $\varepsilon\leqslant\xi\leqslant x$. Now since

$$\lim_{\varepsilon\to 0}\int_{\varepsilon}^{x}\frac{f(t)}{t^{\delta}}\,dt$$

exists, and is finite, corresponding to an arbitrarily small positive number η, a number x_1 can be found, such that for all values of $x<x_1$,

$$\left| \lim_{\varepsilon\to 0} \phi(x) \right| < \frac{\eta}{3}.$$

Thus

$$\frac{1}{x^\delta}\int_0^x f(t)dt = \lim_{\varepsilon\to 0}\frac{1}{x^\delta}\int_\varepsilon^x f(t)\,dt$$

$$= \lim_{\varepsilon\to 0}\left[\phi(x)-\phi(\xi)+\phi(\xi)\left(\frac{\varepsilon}{x}\right)^\delta\right]$$

$$<\eta.$$

Therefore

$$\lim_{x\to 0}\frac{1}{x^\delta}\int_0^x f(t)\,dt = 0.$$

and the theorem is proved.

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3. J. THOMAE, —'Über die Differenzierbarkeit eines Integrales nach der oberen Grenze'; *Nachrichten der Kön. Gesellschaft d. Wissenschaften Zu Göttingen*, (1893), 696-700.

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A PARTIAL ORDERING IN EUCLIDEAN SPACES $E^n (n \geqslant 1)$

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1. *Introduction* :

To the best of our knowledge in the existing mathematical literature no ordering relation in E^n has been attempted or even of some have been suggested, they have not been turned to any good advantage. In this note we shall define a partial ordering $\leqslant$ in E^n [where E^n is the Euclidean space of n dimensions consisting of all ordered n – tuples $(x_1\ x_2 \ldots \ldots x_n)$ of real numbers]. The resultant simplicity in the proofs of the basic theorems of *Analysis* is worthy of notice. For a more detailed study of E^n $(n \geqslant 1)$ from the point of view of this ordering relation reference may be made to our forthcoming work entitled "Lectures on Ana lysis."

2. *Partial ordering relation* $\leqslant$.

Definitions : (1) Two points $a = (a_1, a_2, \ldots an)$ and $b = (b_1, b_2, \ldots bn)$ will be said to have the relation $a \leqslant b$ if and only if $a_i \leqslant b_i$ for each i ;

(2) $a < b$ if and only if $a_i < b_i$ for each i ; and

(3) $a = b$ if now only if $a_i = b_i$ for each i.

It is easy to see that $\leqslant$ constitutes a complete ordering of E^1, the linear continuum ; but for E^n $(n > 1)$ $\leqslant$ is only a partial ordering. For example, $(1, 2, \ldots n)$ and $(2, 3, \ldots n, 1)$ are not related by $\leqslant$. It is also worthy of notice that in E^1 $\leqslant$ is equivalent to $<$ or $=$; while it is not so in E^n $(n > 1)$. For example, if $a = (1, 2, 3, \ldots n)$ and $b = (1, 3, 5, \ldots, n+1)$, neither $a < b$ nor $a = b$ but all the same $a \leqslant b$. If $a \leqslant b$ and $a \neq b$, we shall say that *b exceeds a*.

3. We now turn to the concept of supremum (i. e. the least upper bound) and infimum (i.e. the greatest lower bound) for sets of points in E^n $(n \geqslant, 1)$. Familiarity with these concepts and their properties in E^1 will be assumed.

Let S be any set of points in a fixed Euclidean space E^n $(n \geqslant 1)$. Now if there exists a point α of E^n such that every point of $S \leqslant \alpha$ then S is said to be *bounded above* and any such α is an *upper bound* of S similar definition for S being *bounded below* etc. may be framed. Quite naturally when S is both bounded above and below it is said to be *bounded.* Now if for a given set S there exists a point α such that.

(i) α is an upper bound of S i.e. every point of $S \leqslant \alpha$; and

(ii) no other point β exists such that $\beta \leqslant \alpha$ and every point of $S \leqslant \beta$, then we say that a supremum of S (sup S) exists and $\alpha = \sup S$.

Remark : In E^1 the above property (ii) reduces to (iii) no point β exists such that $\beta < \alpha$ and every point of $S \leqslant \beta$, which is clearly equivalent to

(iv) If $\beta < \alpha$, there exists a point γ of S such that $\beta < \gamma \leqslant \alpha$ But in $E^n (n > 1)$, (ii) and (iv) are not equivalent.

The truth of this remark for E^2 can be seen from the following example :

Consider the set S consisting of the points on the thick line-segment AB.

Then $\alpha = \sup S$; and if $\beta < \alpha$.

(ii) is true but (iv) does not hold clearly sup S is not unique.

We now prove the following analogue of the completeness theorem of E′

Theorem of completeness : In E^n $(n >, 1)$ if S is a non-void set bounded above, then a sup S exists.

Proof. The proof for E^1 is supposed to be well-known. Hence we prove the case for $n > 1$. For this let $\alpha_1 = \operatorname{Sup} S_1$ where S_i is the set of real numbers constituting the ith coordinates of points of S $(i = 1, 2, \ldots \ldots n)$. We claim that

$\alpha = (\alpha_1, \alpha_2, \ldots \ldots \alpha_n)$ is a Sup S. That α is an upper bound of S is quite obvious. Next, if possible suppose that another point $\beta (\leqslant \alpha)$ also is an upper bound. Then for at least one i, $\beta i < \alpha i$. Hence from the property of supremum in E^1 there will be a point c in S such that $\beta_i < c_i \leqslant \alpha i$ and consequently it is not true that $c \leqslant^i \beta$. This is a contradiction with our supposition that β is also an upper bound. Thus α is a Sup S.

4. We wish to conclude this note with proving the Weierstrass-Bolzano theorem for E^n ($n \geqslant 1$). But before we give the proof we make the following definitions.

If $a < b$, then the set of points x such that $a < x < b$ is said to constitute an open *internal* (a, b) in E^n. Similarly the set of points x such that $a \leqslant x \leqslant b$ is a *closed interval* $[a, b]$ in E^n.

An open interval (a, b) containing a point α is said to be a neighbourhood of α If every neighbourhood of a point α contains infinity of points of a given set S, then α is said to be a limiting point of S.

Weierstrass-Bolzano Theorem : In E^n ($n \geqslant, 1$) every bounded infinite set has a limiting point.

Notation :—If S is a set of points in E^n, S_i will denote the set of numbers constituting the *i*th co-ordinates of the point in S.

Now Since S is bounded, let a and b be any two points such that $a < x < b$ for every x in S. Let T be the class of all points c such that $x_1 \leqslant c_1$ for almost finite number of the points x of S. T is clearly non-void, a being in T and it is also bounded above for b is an upper bound. Hence applying the theorem of completeness for E^1, let $\alpha_1 = \text{Sup } T_1$. Clearly any neighbourhood $(\alpha_1 - t, \alpha_1 + t)$ contains infinty of S_1. Thus infinite subset W of S exists such that for infinity of points of W the first co-ordinates lie in $(\alpha_1 - t, \alpha_1 + t)$. Next subjecting W for the second co-ordinates to the same treatment as we did for T as regards first, and continuing this process we get a point $\alpha = (\alpha_1, \alpha_2 \ldots \alpha_n)$ any neighbourhood of which will contain infinity of points of S. Thus α is a limiting point of S.

A NOTE ON THE CLASSICAL DYNAMICS OF TWO INTERACTING SYSTEMS

By S. K. ROY

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Consider a system composed of two parts A and B interacting with each other. It is assumed that the Hamiltonian of the system can be written as

$$K = H_A + H_B + H_{AB} \tag{1}$$

where H_A and H_B are the Hamiltonians of the separate parts and H_{AB} is the interaction term. H_A does not contain any coordinates of the part B, nor are any coordinates of part A involved in H_B. H_{AB} contains the coordinates of both parts, describing the interaction.

The object of this note is to show that a unified picture of motion of the system as described by (1) can be derived in two steps. The first step is to solve the problem without interaction; and the second is to set up a Hamilton-Jacobi type of differential equation in which the Hamiltonian is the interaction part of the Hamiltonian of the system. The solution of this differential equation gives us a generating function of a canonical transformation, the knowledge of which combined with solutions of the system without interaction, completely solves the problem. This approach to the problem, which is generally known as the "Interaction Picture of Motion" is used in quantum field theories and was first introduced by Stückelberg.

If we neglect the interaction term H_{AB} in (1), the problem becomes very simple. We have two non-interacting systems A and B. We define the Lagrangian of the two parts by $L_A(q_A, \dot{q}_A)$ and $L_B(q_B, \dot{q}_B)$ where q_A, q_B are the generic symbols for all coordinates involved in A and B and where the dot denotes differentiation with respect to time t. In terms of these Lagrangians, the Hamiltonians of the two systems A and B can be written separately as H_A and H_B, where

$$H_n(q_n, p_n) = \Sigma p_n \dot{q}_n - L_n(q_n, \dot{q}_n)$$

$$\text{with } p_n = \frac{\partial L_n}{\partial \dot{q}_n}, \quad n = A, B. \tag{2}$$

The Hamilton equations of motion for the two systems are

$$\dot{p}_n = [H_n, p_n], \dot{q}_n = [H_n, q_n] \quad n = A, B. \tag{3}$$

where we have used Poisson's bracket notation. These four equations completely describe the two non-interacting systems.

We introduce now a canonical transformation $q_n \to q_n'$, $p_n \to p_n'$ such that the transformed Hamiltonians H_n' do not involve any q_n'. Then the p_n' are all constants of motion. We may, further, choose the time variation of the generating function G_n of the canonical transformation so as to make the new Hamiltonians vanish. The q_n' are also then constants of motion. This is achieved by finding generating functions G_n, which satisfy the Hamilton-Jacobi equations:

$$H_n \left(\frac{\partial G_n}{\partial q_n}, q_n \right) + \frac{\partial}{\partial t} G_n (q_n, q_n', t) = 0. \tag{4}$$

In the above equations the G_n are functions of q_n, $p_n' = a_n$ (which are constants) and t. The q_n' are derived from

$$q_n' = \frac{\partial G_n}{\partial a_n} \tag{5}$$

The time derivatives of q_n' are then given by $\dot{q}_n' = \partial H_n / \partial p_n' = 0$, i. e. q_n' are constants $= b_n$. The generating functions G_n's are then given by

$$G_n = G_n (q_n, b_n, t) \tag{6}$$

The equations of transformation

$$p_n = \frac{\partial G_n (q_n, q'_n, t)}{\partial q_n}, \tag{7}$$

$$p'_n = - \frac{\partial G_n (q_n, q'_n, t)}{\partial q'_n}$$

give the p_n and q_n as functions of p_n and q'_n which are constants :

$$p_n = p_n(p'_n, q'_n, t)$$

$$q_n = q_n (p'_n, q'_n, t) \tag{8}$$

There are infinitely many such solutions. The particular solutions in which the p'_n and q'_n are the initial values of the moments and co-ordinates are of significance.

In the case when the Hamiltonians of the systems A and B do not involve the time t explicitly, simpler Hamilton-Jacobi equations may be formed. For $H_n \left(\frac{\partial G_n}{\partial q_n}, q_n \right)$ not being dependent explicitly on time t, it is possible to chose a function $W_n (q_n, q'_n)$ such that

$$G_n (q_n, q_n', t) = W_n (q_n, q_n') - E_n t \tag{9}$$

where the E_n are constant. The Hamilton-Jacobi equations then reduce to

$$H_n\left(\frac{\partial W_n}{\partial q_n}, q_n\right)=E_n. \tag{10}$$

The constants E_n are naturally found to be the energies of the component systems.

A unified picture of the combined system A and B (without interaction) can be given by defining :

$$\begin{aligned} L &= L_A + L_B \\ H &= H_A + H_B \\ G &= G_A + G_B \\ W &= W_A + W_B \\ E &= E_A + E_B \end{aligned} \tag{11}$$

$$\begin{gathered} \dot{p}_n = [H, p_n],\ \dot{q}_n = [H, q_n] \qquad n = A, B \\ H\left(\frac{\partial G}{\partial q_n}, q_n\right) + \frac{\partial G}{\partial t} = 0 \\ H\left(\frac{\partial W}{\partial q_n}, q_n\right) = E_A + E_B = E. \end{gathered} \tag{12}$$

Now let us consider the interaction term also. In this case we shall represent the coordinates by Q_n and P_n. The Lagrangian L^I is given by—

$$L^I = L_A(Q_A, \dot{Q}_A) + L_B(Q_B, \dot{Q}_B) + L_{AB}(Q_A, Q_B, \dot{Q}_A, \dot{Q}_B) \tag{13}$$

where L_A and L_B are the same functions as before. The Hamiltonian K is given by—

$$K = \sum_{n=A,B} \frac{\partial L^I}{\partial \dot{Q}_n} \cdot \dot{Q}_n - L^I \tag{14}$$

$$= H(P_n, Q_n) + H_{Int} \tag{15}$$

where
$$H_{Int} = \sum_{n=A,B} \frac{\partial L_{AB}}{\partial \dot{Q}_n} \cdot \dot{Q}_n - L_{AB} \tag{16}$$

and
$$P_n = \frac{\partial L^I}{\partial \dot{Q}_n} \tag{17}$$

The Hamilton equations of motion are

$$\dot{P}_n = -\frac{\partial K}{\partial Q_n},\ \dot{Q}_n = \frac{\partial K}{\partial P_n}. \tag{18}$$

Let us suppose that we know the solutions of the system when the interaction is zero. We represent these solutions without interaction by—

$$p_n = p_n(p_n', q_n', t)$$
$$q_n = q_n(p_n', p_n', t)$$
$$n = A,B$$

where p', q''s are constants, and

$$p_n = \frac{\partial G_n}{\partial q_n},\ p_n' = -\frac{\partial G_n}{\partial q_n'},\ G_n = G_n(q_n, q_n', t). \tag{20}$$

Now we introduce a canonical transformation from the variable $Q, P \rightarrow Q^I, P^I$ such that the new Hamiltonian $K^I(Q^I, p^I)$ to which K is transformed has the same functional dependence on $Q_n{}^I$, $P_n{}^I$ as $H(p_n, q_n)$ has on p_n, q_n. In other words, we seek a generating function $G^I(Q^I, Q_n{}^I, t)$ given by the Hamilton-Jacobi equation—

$$K\left(\frac{\partial G^I}{\partial Q_n},\ Q_n\right) + \frac{\partial G^I}{\partial t} = K^I\left(-\frac{\partial G^I}{\partial Q^I_n},\ Q_n{}^I\right), \tag{21}$$

such that $K^I(P^I, Q^I) = H^I(P^I, Q^I)$, (22)
where H^I is obtained from the Hamiltonian $H(p, q)$ of the system without interaction by replacing p, q by P^I, Q^I.

The Hamilton equations of motion in these variables are—

$$P_n{}^I = -\frac{\partial K^I}{\partial Q_n} = -\frac{\partial H^I}{\partial Q_n},\ Q_n{}^I = \frac{\partial K^I}{\partial P_n} = \frac{\partial H_I}{\partial P_n}, \tag{23}$$

which show that $P_n{}^I$ and $Q_n{}^I$'s are given by exactly the same equations which give p_n, q_n. Thus the problem of solving (23) is reduced to finding a generating function $G^I(Q_n{}^I\ Q_n{}^I, t) = G^I(Q_n, q_n, t)$ from equation (21).

The knowledge of the function G^I gives the transformation $(P_n, Q_n, t) \rightarrow (P_n{}^I, Q_n{}^I\ t) = (p_k, p_k, t)$ and by means of transformation $G(q_n, q_n', t)$ we know that $(p_n, q_n, t) \rightarrow (p_n', q_n', t)$, and therefore we have—

$$P_n = P_n(p_n', q_n', t) \tag{24}$$
$$Q_n = Q_n(p_n', q'_n, t)$$

The two transformations given by the generating functions G^I and G can be combined into one $J(Q, q^1, t)$ which will transform directly from variables (P_n, Q_n, t) to (p_n', q'_n, t). It can be shown that—

$$J(Q_n, q_n', t) = G^I(Q_n, q_n, t) + G(q_n, q'_n, t) \tag{25}$$

If the Hamiltonians H and K are time-independent, the generating functions G, G^I and J can be written as—

$$
\begin{aligned}
G &= W(q_n, q_n') - E\,t,\\
G^I &= W^I(Q_n, q_n) - E^I\,t,\\
J &= W^J(Q_n, q'_n) - E^J\,t,
\end{aligned}
\tag{26}
$$

where all the E's are constants. It can be readily shown that—

$$
\begin{aligned}
W^J &= W^I + W\\
\text{and}\qquad E^J &= E^I + E
\end{aligned}
\tag{27}
$$

It may be noticed that E is the energy of the two parts without interaction as given by (11), E^J the total energy with interaction, and E^I the energy of interaction.

The solution of the problem of the interacting system is therefore achieved in two steps. The solution of the problem without interaction, which is given by—

$$\dot{P}_n = -\frac{\partial H}{\partial q}, \quad \dot{q}_n = \frac{\partial H}{\partial p_n},$$

leads to (19). Then the solution of (21) completes the problem. In fact G is to be found from the equation (21) which is equivalent to—

$$K\left(\frac{\partial G^I}{\partial Q_n}, Q_n\right) - H\left(-\frac{\partial G^I}{\partial q}, q_n\right) + \frac{\partial G^I}{\partial t} = 0. \tag{28}$$

This equation together with $G(Q, q, t) = W^I(Q, q) - E^I t$ reduces to—

$$K\left(\frac{\partial W^I}{\partial Q_n}, Q_n\right) - H\left(-\frac{\partial W^I}{\partial q_n}, q_n\right) = E^I \tag{29}$$

The equation (29) depends only on the interaction part of the Hamiltonian K – H.

TWO DIOPHANTINE PROBLEMS

By

Dr. A MOESSNER (GUNZENHAUSEN, GERMANY)

and

T. N. SINHA (T. N. J. College, Bhagalpur)

I. The equation $A^{2n}+B^{2n}=C^{2n}+D^{2n}$ ... (1)

For $n=1$, the general solution of the aforesaid equation is already known and particularly for those cases also where $A=B$ or $C=D$, or where one of the bases A, B, C, D is equal to 0. Deviating from the usual methods of solving the given equation for $n=1$, we proceed with the relation $x \,.\, y=u \,.\, v$. If there is such a relation then we have also $(x+y)^2+(u-v)^2=(x-y)^2+(u+v)^2$

Thus the solutions of the given equation (1) for $n=1$ are given by $A=x+y$, $B=u-v$, $C=x-y$, $D=u+v$ where $x.\,y=u.\,v$.

Example : $8.3=6.4$ gives $(8+3)^2+(6-4)^2=(8-3)^2+(6+4)^2$ or $11^2+2^2=5^2+10^2$.

For solving the given equation (1) for $n=2$ we make use of the same relation and put

$$x+y=A^2,\ x-y=C^2,\ u-v=B^2,\ u+v=D^2$$

Example : $x=10585$, $y=7104$, $u=21460$, $v=3504$ where $10585.\,7104=21440.\,3504$ gives $133^4+134^4=59^4+158^4$. One observes that if $n>1$, the aforesaid equation is rationally impossible for those cases $A=B$ or $C=D$ or where one of the bases A, B, C, D is equal to 0*

II. The relation

(*i*) $F_1,\ E_2,\ldots,\ E_r \overset{K}{=} F_1,\ F_2,\ldots F_r$ for $K=0,\ 1,\ 2,\ldots n$

with $V=\dfrac{2\left(\Sigma F_i^{\,n+2}-\Sigma E_i^{\,n+2}\right)}{(n+2)\left(\Sigma F_i^{\,n+1}-\Sigma E_i^{\,n+2}\right)}$ gives

(*a*) when n is even

$E_1,\ E_2,\ldots E_r,\ V-E_1,\ V-E_2\ldots V-E_r \overset{K}{=} F_1,\ F_2,\ldots F_r,\ V-F_1,\ V-F_2,\ldots,\ V-F_r$ for $K=0,\ 1,\ 2,\ \ldots n,\ n+1,\ n+2,\ n+3.$

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*Fermat's last theorem : $A^n+B^n=C^n$ is impossible in integers if $n>2$, which, however, has not yet been proved in all its generality.

(*b*) when n is odd

$$E_1, E_2, \ldots, E_r, V-F_1, V-F_2, \ldots V-F_r \overset{K}{=} F_1, F_2, \ldots, F_r, V-E_1,$$
$$V-E_2, \ldots, V-E_r \text{ for } K=0, 1, 2, \ldots n, n+1, n+2, n+3.$$

We apply (*a*) of this theorem to the relation

(*ii*) $E_1^r, E_2^r \overset{K}{=} F_1^r, F_2^r$ for $K=0$ where according to (*i*)

$$V = \frac{F^{2r}+F_2^{2r}-E_1^{2r}-E_2^{2r}}{F_1^r+F_2^r-E_1^r-E_2^r}$$

We then get the system of equations of the form

$$(iii) \quad \begin{cases} E_1^r+E_2^r+H+L=F_1^r+F_2^r+M+P \\ E_1^{2r}+E_2^{2r}+H^2+L^2=F_1^{2r}+F_2^{2r}+F_2^{2r}+M^2+P^2 \\ E_1^{3r}+E_2^{3r}+H^3+L^3=F_1^{3r}+F_2^{3r}+M^3+P^3 \end{cases}$$

Example : We put $r=5$ and, in order to get whole numbers, $E_1=690$, $E_2=1380$, $F_1=345$, $F_2=1725$ so that

$(690^5), (1380^5) \overset{K}{=} (345^5), (1725^5)$ for $K=0$. Thus we get the following numerical solution for (*iii*)

$$690^5+1380^5+H+L=345^5+1725^5+M+P$$
$$619^{10}+1380^{10}+H^2+L^2=345^{10}+1725^{10}+M^2+P^2$$
$$690^{15}+1380^{15}+H^3+L^3=345^{15}+1725^{15}+M^3+P^3$$

where $H=15\ 575\ 018\ 014\ 569\ 375$, $L=20\ 423\ 515\ 196\ 469\ 375$

$M=5\ 306\ 174\ 688\ 791\ 250$, $P=20\ 575\ 030\ 733\ 403\ 750$.

$\mathscr{S}$ FUNCTIONAL NUMBERS AND THEIR ORDER EXTENSION

By

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Starting with ordinal numbers we have constructed two distinct systems of ordinal real numbers Ron (w_u) and $\dagger_1$. *Ron* (w^θ), where On (w_u) is the class of all ordinal numbers $< w_u$, while On (w^θ) is the class of all ordinal numbers $< w^\theta$, θ being an indecomposable ordinal number and further ordinal sum and ordinal product are definable in On (w_u) whereas natural sum and natural product in On (w^θ). But neither *Ron* (w_u) nor *Ron* (w^θ) is rich enough as to determine the order of greatness or order of smallness of Logarithmic Exponential functions shortly L functions.

Thomae,[1] Pincherle[2] and Borel[3] have also attempted to represent the orders of L functions by their symbols. But little application has yet been found for their systems of notations.

We thus feel a necessity to construct altogether a new system of numbers with a view to determine the order of greatness of Logarithmico Exponential functions.

$\dagger_1$ The author's two papers entitled

"(a) Ordinal Integers, Rational and Real Numbers.

(b) Generalisation of ordinal Integers, Rational and Real Numbers." are under publication.

[1] J. Thomae Elementare Theorie der analytische Functionen einer complex Veranderlichen (Second Edition Haele Nebert 1898) 144.

[2] S. Pincherle Alcune Osservazioni Sugli Ordini des infinite delle funzioni Memorie della, R. Accademia della Scienzedi Bologona (4, 5, 1884, 739-750).

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E. Borel : Lecons Surla theorie dela Groissance (Paris Ganthie Villars 1910) p. 14.

Here we propose first the construction of an ordered additive system of functional numbers and next to find its order extension.

1. L. Functional Numbers.

With the aid of the following preliminary dfns, theory of L†[2] functional numbers is introduced. Here without any loss of generality, we will consider the constructions from a class of L functions which are either positive, steadily increasing and tend to infinity or are null and steadily decreasing to zero. It is to be noted that functional numbers can also be constructed from a class of negative L functions tending to $-\infty$ as well as from null functions in an analogous manner. Throughout we shall assume that the variables $\alpha, \beta, \gamma, S, \ldots$ (with suffixes if needed) range over the class of real numbers (natural).

It is possible to introduce an ordering relation in the domain of L functions.

†[3] Dfn 1. (*i*) $f <' g \equiv f/g \longrightarrow 0,$
(*ii*) $f >' g \;.\!\equiv\!.\; g <' f$

Theorem 1.

(*i*) $f >' g \longleftrightarrow f/g \longrightarrow \infty$

For if $g \longrightarrow 0$, then $f/g \longrightarrow \infty$

(ii) Null (f) $<'$ Pos (g)

The proof follows from the dfn.

Next an equivalence relation $\simeq'$ is defined in the domain of L functions.

Dfn. 2. $f \simeq' g \;.\!\equiv\!.\; f/g \to \alpha \,.\, 0 < \alpha < \infty$

that is, $f \simeq' g$ if and only if $f/g \to \alpha$

where α is a positive real number.

N.B. (i) A null function $\simeq'$ a positive function.

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†[2] An L function is a real one valued function defined by a finite combination of the functional symbols α (.......), e (.......) log (.......), α is a positive real number (natural), operating on the variable x and the number constants, where of course the value of x may be restricted to greater than some definite value, say x_0.

†[3] If in the domain of positive and null L functions, negative L functions are included, then we may assume that

(*i*) neg $f_1 <'$ null $f_2 <'$ Posf_3

(*ii*) neg $f_1 <'$ neg $f_2 \longleftrightarrow$ Pos $f_1 >'$ Pos f_2

(ii) $f <' g \longrightarrow f + g \simeq' g$

It is then readily seen that $\simeq'$ is reflexive, symmetric and transitive.

Dfn. 3. A class of functions $[f]$ defined by the relation $\simeq$ is said to be a functional number.

Functional numbers are positive (Pos), negative (neg), null, finite (fint), transfinite (trf) and infinitesimal (Inf) as defined below.

Without any loss of generality let us assume that the variable $x \to \infty$ unless otherwise stated.

Dfn. 4. (i) Pos $[f] \;.\equiv.\; f \longrightarrow \infty$

(ii) Neg $[f] \;.\equiv.\; f \longrightarrow 0$

(iii) Null $[f] \;.\equiv.\; C(f)$,

that is, $[f]$ is said to be a positive or a negative or a null functional number acc. as $f \to \infty$ or $f \to 0$ or f is a constant function

N. B :— A null functional number is the class of constant functions and is denoted by $[c]$

Dfn. 5. (i) Fint $[f] \;.\equiv.\; \exists f_1 \cdot \exists \alpha \,[f_1 \in [f] \cdot f_1 = x^{\alpha}]$,

i. e., $[f]$ is said to be a finite functional number if and only if there exists a function f_1 belonging to $[f]$ such that $f_1 = x^{\alpha}$.

(ii) Inf. $[f] \;.\equiv.\; \exists f_1 \,[f_1 \in [f] \cdot f_1 = l_r x \cdot r \neq 0]$

that is, $[f]$ is said to be an infinitesimal functional number if and only if there exists a function f_1 belonging to $[f]$ such that $f_1 = l_r x$ where $r \neq 0$ (where $l_r x$ is r times repeated logarithimic function)

(iii) trf. $[f] \;.\equiv.\; \sim$ fint $[f] \;.\; \sim$ Inf. $[f]$,

that is, $[f]$ is said to be a transfinite functional number if it is neither a finite nor an infinitesimal.

Let us denote the class of finite, transfinite and infinitesimal functional numbers by $\mathscr{L}$.

An ordering relation as well as addition is introduced in the domain of $\mathscr{L}$.

Dfn. 6. (i) $[f] < [g] \;.\equiv.\; f <^1 g$

(ii) $[f] > [g] \;.\equiv.\; [g] < [f]$

Theorem 2. $\mathscr{L}$ is simply ordered.

For if f and g are two numbers then either $[f] = [g]$, or $[f] > [g]$ or $[f] < [g]$

Theorem 3. $\mathscr{L}$ is dense, that is between any two functional numbers there always lies a third element between them.

Proof : Let $[f]$ and $[g]$ are the two numbers and for definiteness let us assume that $[f] < [g]$.

Clearly there exists a $[\sqrt{fg}]$ such that $[f]<[\sqrt{fg}]<[g]$, for it can easily be proved that $f <' \sqrt{fg} <' g$ as follows :—

$$f/\sqrt{fg}=(f^2/fg)^{\frac{1}{2}}=(f/g)^{\frac{1}{2}}\longrightarrow 0$$

Also $\sqrt{fg}/g=(fg/g^2)^{\frac{1}{2}}=(f/g)^{\frac{1}{2}}\longrightarrow 0$ as $f<' g$

Dfn 7. $[f]+[g] \,.\!\equiv\!.\, [f.g]$

Properties of $+$

P 1. $+$ is commutative for in the definition $fg=gf$

P 2. $+$ is associative for $f.(gh)=(fg).h$

P 3. $+$ admits $[c]$ as additive identity element, for

$$[f]+[c]=[f.c]\simeq[f]$$

P 4. $+$ admits iniverse elements, i.e.,

$$[f]\to[1/f]\ [\,[f]+[1/f]=[1/f]+[f]=[c]\,]$$

It is to be noted that $+$ has all the usual properties of a group.

Theorem 3. (i) $[f]-[g]=[f/g]$

(ii) $-[f]=[1/f]$ by P. 4

Generally the product of two functional numbers does not exist. However both sided product of a real number and a functional number is definable.

Dfn 8. (i) $\mathscr{L}.\,[f]=[f^{\alpha}]$

(ii) $[f]\,.\,\alpha=[f^{(\alpha)}]$

Dfn 9. $|\,[f]\,|=[f] \,.\!\equiv\!.\, f\longrightarrow\infty$

$=[1/f] \,.\!\equiv\!.\, f\longrightarrow 0,$

The system of real numbers (natural) can be mapped on a part of $\mathscr{L}$ by a correspondence which is one to one, similar and isomorphic with respect to $<$ and $+$.

The following correspondence defines sum on isomorphism.

$$\alpha\longleftrightarrow[x^{\alpha}]$$

An Interpretation of L functional numbers. Finally we shall give an interpretation of L functional numbers. $[f]$ is interpreted as the order of greatness of f, i. e.,

$$[f]=O(f).$$

It is to be noted that $|[f]|$, the absolute value of $[f]$ is sometimes said to be the order of greatness or order of smallness of f according as $[f]$ is positive or negative.

It has already been stated that fint $[f]$ is associated with a real number by a correspondence which is one to one, similar and isomorphic w.r.t. $<$ & $+$.

Thus it is possible to represent $O(x^{\alpha})$ by the real number α. But a trf $[f]$ as well as in f [f] can not be represented by an ordinal real number neither in the domain of Ron $(w\mu)$ nor in Ron (w^{θ}).

For if possible $O(e^{x})$ is represented by the smallest transfinite ordinal number w since $e^{x} >' x^{n}$ for all the natural numbers n.

Then clearly $1/n.\ [e^{x}]$ $\left[e^{\frac{x}{n}} \right]$

and $[e^{x}].\ \frac{1}{n} = \left[e^{x \frac{1}{n}} \right]$ and may respectively be represented by $\frac{1}{n}.\ w$ and $w.\ \frac{1}{n}$.

But in the domain of Ron (w_{μ}), $\frac{1}{n}.\ w = w$ and thus $e^{\frac{x}{n}} \simeq' e^{x}$. which is absurd.

While $w.\frac{1}{n}$ is not at all definable.

Also in Ron (w^{θ}), $\frac{1}{n}.\ w = w.\ \frac{1}{n} e^{\frac{x}{n}} \simeq' e^{x \frac{1}{n}}$. which is again absurd.*

* The order extension of $\mathscr{L}$ will be considered in the next issue of the Journal.

A NEW APPROACH TO THE THEORY OF GRAVITATION

By

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Long before Newton fomulated the law of Universal gravitation the secret of gravitation was known to many philosophers who held it as things must have their own places. The smoke went up which was its proper position and heavy particles went down because of its proper place. It was Galileo who proved experimentally that in vacuum all things, heavy as well as light, fall to the earth simultaneously due to the gravity. The well known astronomer Keppler studied the motion of heavenly bodies thoroughly and formulated the laws known after his name. These laws were found to be surprisingly accurate. And celestial mechanics originated by Tycho Brahe and developed by Keppler and others was further confirmed by the help of astronomical telescope designed by Galileo and others +. And in his turn Newton formulated the law of Universal gravitation. The amount and direction of the force of attraction between two heavenly bodies is very accurately given by the law. But its nature was not quite clear. Why should two bodies attract each other? The law of universal gravitation as expounded by Newton tells us nothing. And so it remains unanswered uptill now. Moreover there are facts which can not be explained by Newton's law of attraction. So now and then scholars working in this branch of Physics have brought forward theories claiming to explain these difficulties but so far only Einstein's law of General Relativity can claim to be nearer the truth. Still it has not been able to clear all the anomalies of the nature, encountered by Newton's law.

It will be the purpose of this article to touch the two theories—Newton's and then that of Einstein's—and come to their difficulties. And lastly to suggest a way out as far as possible. I must confess at the very outset that I have not got the solution. Only a new approach to the problem has been suggested.

Newton's law of Gravitation says as everybody knows—"Every particle of matter in the universe attracts every other particle at all distances in the line joining them with a force which varies directly as the product of two masses and inversely as the square of the distance between them". It is evident that Newton thought that action of one body on another proceeded with infinite velocity; which is obviously wrong. Secondly, Newton thought the mass of a body to be constant independent of its state of motion ; which also is now found to be untenable. Also as previously stated Newton never explained the cause of attraction between bodies. Finally it must be mentioned that Newton's law is ambiguous and so it is difficult to point out its discrepancies adequately. And he was not very satisfied with it. (1). In addition to all these, there were some facts such as the anomalous motions of Jupiter, Saturn and the Moon which the law could not explain. And so "all the great mathematicians of the continent—Hygens in Holland, Leibnitz in Germany, Johann Bernoulli in Switzerland, Cassimi in France—rejected the Newtonian theory altogether." (2).

So a stage was reached which has been nicely described in these lines "if the machine (mathematical theory) works successfully correct predications of events can be made under circumstances properly controlled. As more and more knowledge is gained in a particular field of phenomena a stage soon comes when the mathematical machine fails. A new machine has now to be devised." (3).

A bold and fundamental step was taken in this direction by Albert Einstein. In his attempt he was guided by these facts:—

1. Gravitational fields have a basic proper that all bodies move in them in the same manner, independently of mass or charge; provided the initial conditions are the same. This is illustrated from the laws of free fall in the gravity field of the earth. Here all bodies acquire one and the same acceleration irrespective of their masses if they start with the same initial velocity. Also it was proved by accurate and rigorous experiments that the inertial and gravitational masses of the same particle were equal. Inertial mass may be determined by measuring the force and acceleration acquired by the particle as result of this force. Whereas gravitational mass is found out by the formula deduced from the Newtonian law of gravitation. The physical nature of these masses is quite different. Inspite of this their quantitative significance is always the same.

2. For small velocities Newton's law is amazingly true. Therefore any law for the field of gravitation must change into Newton's law for small velocities.

3. The principle of covariance, which states that the laws of physics can be expressed in a form which is independent of the co-ordinate system used.

On these supports Einstein built his brilliant theory of General Relativity, which along with the quantum mechanics form the greatest achievements of the modern civilization.

The theory of General Relativity was supposed to be confirmed by many important relativists like A. S. Eddintion. The well known three experiments often cited as proving the predictions of the theory are (a) deflection of light by the gravitational field of a star, (b) the influence of the gravitational potential on the frequency of emitted light, (c) slow rotation of elliptic circuits of planets.

The theory also explained the expanding motion of galactic systems. (4).

Let us examine the hypotheses one by one. Einstein himself concluded that principle of equivalence holds good only for a very limited part of universe. That is to say the fields to which non-intertial reference systems are equivalent are not completely identical with actual gravitational fields. At infinite distances from the gravitating bodies, the actual gravitational fields go to zero. On the contrary the fields which are equivalent to non-inertial frames of reference increase without limit at infinity ; or in any case remain finite in value. Therefore we conclude that it is impossible by any choice of reference frame to eliminate an actual field since it vanishes at infinity.

Further Einstein's equation of gravitation – $R_{ik} = \frac{8\pi k}{c^4}(T_{ik} - \frac{1}{2}g_{ik}T)$ proves to be so complex that they have to be simplified in order to get definite conclusions. And even then they can be applied for a finite region of the universe.

Regarding the three experimental verifications of the theory it has been pointed out by Prof. V. V. Narlikar that in 1947 the amount of deflection of a ray of light grazing sun's limb is found to be $2''.01 \pm 0''.27$ seconds whereas according to the theory it should be $1''.75$ seconds. The error being much more than allowable experimental error, there are men like Freundlich who still believe that a satisfactory quantitative agreement between theory and experiment on the subject has not been established. (5).

The second is the observational test of the anomalous motion of the perihelion of Mercury. This is of course in agreement with Einstein's theory. But other alternative explanations of this phenomena is not difficult to cite. (5),

Similarly third proposed test regarding the displacement of the red spectral lines emitted in a strong gravitational field was explained before General Relativity was discovered. Further information regarding these can be obtained in the article of E. Wiechert, Phys, ZSXVII, 1916 Page 44?.

As earlier stated Friedman proved that according to Einstein's equation the universe must be expanding. And it was confirmed by astronomical observations. But Sri K. P. Stanyukovich observes "there is no reason why phenomena observed in the visible part of the universe should be attributed to the whole of the boundless cosmos. Our galaxy and other known contiguous galaxies evidently lie in that part of the metagalaxy which is expanding at present. But it does not follow that the whole of the universe is expanding ; an episode can not be taken for the whole of the picture. (7).

The general theory of relativity tried to generalize the postulates of the special theory. But Sri V. Fock observes that the generalization has not been sufficiently correct. The same word relativity has two different meanings in the two theories. In the special theory it means uniformity and in the general theory it is used for covariance. And as covariance has nothing to do with uniformity, there is a great confusion. This confusion leads to statements like "rotation is relative" which is obviously false. (8).*

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