Bihar Mathematical Society

TSTM Examination (Senior) Solution 2019

Full Marks -100 **Time: $2\frac{1}{2}$ Hours**

Answer all questions. All questions carry equal marks.

1. **Let a and b be positive real numbers and $a\sqrt{a}+b\sqrt{b}=183,\ a\sqrt{b}+b\sqrt{a}=182.$ Find $\frac{9}{5}(a+b)$.**

   मानलिया a तथा b वास्तविक धनात्मक संख्यायें हैं, तथा $a\sqrt{a}+b\sqrt{b}=183,\ a\sqrt{b}+b\sqrt{a}=182.$ तो $\frac{9}{5}(a+b)$ का मान निकालें।

   Solution. Let $a=p^2$and $b=q^2$ so that the given equations

   Now $(p+q)^3=p^3+q^3+3pq(p+q)=183+546=729$

   $\Rightarrow p+q=9.$

   $\Rightarrow pq(p+q)=182$

   $\Rightarrow pq=\frac{182}{9}$

   Now $(a+b)=p^2+q^2=(p+q)^2-2pq=81-\frac{364}{9}$

   $=\frac{729-364}{9}=\frac{365}{9}$

   $\Rightarrow \frac{9}{5}(a+b)=\frac{9}{5}\times\frac{365}{9}=73$

2. **If $2^x=3^y=6^{-z}$, then find the value of $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}$.**

   यदि $2^x=3^y=6^{-z}$, तो $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}$ का मान निकालें।

- **Solution:** If $2^x=3^y=6^{-z}$ then value of $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}$.

  Let $2^x=3^y=6^{-z}=k$(say)

  $\therefore 2=k^{\frac{1}{x}}, 3=k^{\frac{1}{y}}$ and $6=k^{-\frac{1}{z}}$

  $\therefore$ here $2\times 3=6$

  $\therefore k^{\frac{1}{x}}\times k^{\frac{1}{y}}=k^{-\frac{1}{z}}$

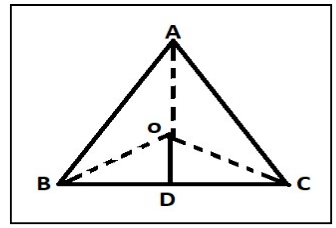
  $\Rightarrow k^{\frac{1}{x}+\frac{1}{y}}=k^{-\frac{1}{z}}$

  $\Rightarrow \frac{1}{x}+\frac{1}{y}=-\frac{1}{z}$

  So, $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=0$

3. **Let O be the incentre of a triangle ABC and D be a point on the side BC such that OD⊥ BC. If $\angle BOD = 15^0$, then find $\angle ABC$.**
मानलिया **O** त्रिभुज ABC का अन्तःकेन्द्र है तथा भुजा BC पर D एक ऐसा विन्दु है कि **OD⊥ BC.** यदि $\angle BOD = 15^0$,तो $\angle ABC$ मान निकालें।

Solution: we know that,



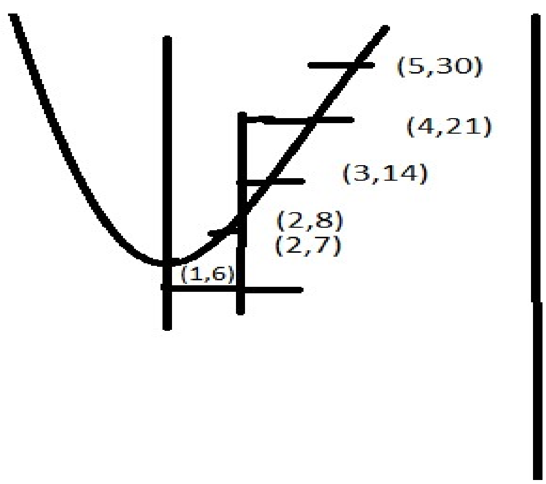
O is the any bisector of all three sides, in incentre. $OD \perp BC$

$\therefore \angle BDO = 90^0$ and $\angle BOD = 15^0$ (given)

$\therefore \angle DBO = 180^0 - (90^0 + 15^0) = 75^0$

$\therefore \angle ABC = 2 \times \angle DBO = 75^0 \times 2 = 150^0$

4. **What is the sum of the squares of the roots of the equation $x^2 - 7[x] + 5 = 0$, where [x] is the maximum integer less than x?**
यदि $[x]$ अधिकतम पूर्णांक हो जो x से कम है तो समीकरण $x^2 - 7[x] + 5 = 0$ के मूलों क वर्गों के योग का मान क्या होगा ?
Solution:



$0 \le x < 1$ $y_1 = x^2 + 5$

$y_2 = 7[x] - 0$

$1 \le x < 2$ $y_1 = x^2 + 5$

$y_2 = 7$

$2 \le x < 3$ $y_1 = x^2 + 5$

$y_2 = 14$

$3 \le x < 4$ $y_1 = x^2 + 5$

$x = \sqrt{2}$, $x = \sqrt{23}$

5. **In a class the numbers of boys and girls are in the ratio 4 : 3. When 8 boys and 14 girls are absent, the number of boys is the square of the number of girls. Find the total number of students in the class.**
एक कक्षा में लड़के एवं लड़कियों की संख्याओं का अनुपात 4 : 3 है। यदि 8 लड़के तथा 14 लड़कियाँ अनुपस्थिति हों तो लड़कों की संख्या लड़कियों की संख्या का वर्ग है। कक्षा में कुल विद्यार्थियों की संख्या ज्ञात करें।
Solution. Let number of boys= $4n$ and number of girls= $3n$
Accordingly
$4n - 8 = (3n - 14)^2 = 9n^2 - 84n + 196$
$\Rightarrow 9n^2 - 84n - 4n + 196 + 8 = 0$

$\Rightarrow 9n^2 - 88n + 204 = 0$

$\Rightarrow n = \frac{88 \pm \sqrt{7744-7344}}{2\times9} = \frac{88\pm20}{2\times9} = \frac{108}{18}, \frac{68}{18}$

$\Rightarrow n = 6, \frac{34}{9}$

Number of students in the class= $4n + 3n = 24 + 18 = 42$

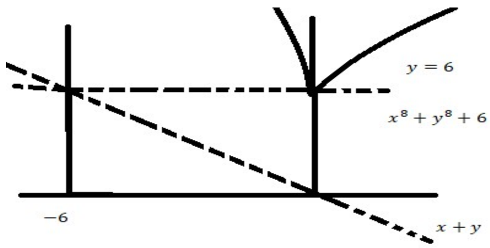
6. **Find all real numbers satisfying $x^8 + y^8 = x + y - 6$.**

$x^8 + y^8 = x + y - 6$ को संतुष्ट कारने वाले सभी वास्तविक संख्याओं को ज्ञात करें।

Solution: $x^8 + y^8 = x + y - 6$

$x^8 + y^8 + 6 = x + y$

$\because x^8 + y^8 + 6 = x + y$



If at all answer solution. It must be in II quadrant and for $x \leq -6$

$\because x^8 + y^8 + 6 \geq 6$

But $y = -x$

$\because x^8 + y^8 + 6$ given

$2(x^8) > 2(-6)^8 + 6$

$\Rightarrow y = -x \therefore$ Number of Real solution exist.

7. **How many six-digit numbers divisible by 25 can be formed using digits 0, 1, 2, 3, 4, 5 without repetition?**

0, 1, 2, 3, 4, 5 को लेकर बिना पुरावृति के छः अंकों की कितनी संख्या बनायी जा सकती है जो 25 से विभाज्य हो?

Solution: Divisibility by 25 → last digit should be zero or five.

If we take 5 as last digit then the possibility will be $4 \times 4 \times 3 \times 2 \times 1 = 96$

If we take zero as last digit then the possibility will be $5! = 120$

Hence total possibility will be $96 + 120 = 216$

8. **Fin the remainder when 19^{92} is divided by 92.**

यदि $\mathbf{19^{92}}$ का $\mathbf{92}$ से भाग दिया जाय तो शेष का मान ज्ञात करें।

Solution. Chinese Remainder theorem (along with other results). First note 92= 4 × 23 with gcd

(4,23) =1. Let us call N= 19^{92}. We will compute, N(mod 4) and N (mod 23) and then use CRT to

compute N (mod 92).

First, N (mod 4) = $(19)^{92}(\,mod\ 4) = (-1)^{92}(mod\ 4) = 1$

and $N(mod\ 23) = 19^4.[(19)^{22}\ (mod\ 23)]^2$

$(mod\ 23) = (-4)^4(mod\ 23) = (16)^4$

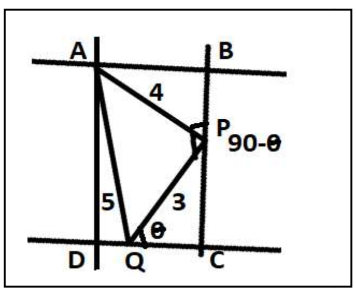
$(mod\ 23) = (-7)^2(mod\ 23) = 49$

$(mod\ 23) = 3.$

Note in the above we have used Fermat's Little Theorem. Now, If you know CRT, you can directly say $N(\,mod\ 92)\ =\ 49.$

If not, you can compute it. One way to do it is write down two lists of numbers (one for each relation) and pick out the first common number.

9. Let $ABCD$ be a square. P and Q are any two points on BC and CD respectively such that $AP = 4\ cm$, $PQ = 3\ cm$, and $AQ = 5\ cm$. Find the side of the square.
मानलिया $\boldsymbol{ABCD}$ एक वर्ग है। $\boldsymbol{P}$ तथा $\boldsymbol{Q}$ क्रमशः $\boldsymbol{BC}$ तथा $\boldsymbol{CD}$ पर दो ऐसे विन्दु है कि $\boldsymbol{AP = 4\ cm, PQ = 3\ cm}$ तथा $\boldsymbol{AQ = 5\ cm}$, वर्ग की भुजा का मान निकालें।



$AB = 4sin\theta$
$\text{PB} = 4\cos\theta$
$PC = 3sin\theta$
$AB = PB \perp PC$
$4sin\theta = 4cos\theta + 3sin\theta$
$sin\theta = 4cos\theta \Rightarrow tan\theta = 4$
$tan\theta = 4 \Rightarrow sin\theta = \frac{4}{\sqrt{117}}$

$\therefore AB$ is side= $4sin\theta = \frac{16}{\sqrt{117}}$

10. If $(a^2 - b^2)sin\theta + 2ab\ cos\theta = a^2 + b^2$, then find $tan\theta$.
यदि $\boldsymbol{(a^2 - b^2)sin\theta + 2ab\ cos\theta = a^2 + b^2}$, तो $\boldsymbol{tan\theta}$ का मान निकालें।

Solution: The coefficient of $sin\theta = perpendicular$(p)

The coefficient of $cos\theta = base$(b)

and constant is *hypotenuse*= h

$\therefore\ tan\theta = \frac{p}{b}$

$\Rightarrow\ tan\theta = \frac{a^2-b^2}{2ab}$