TSTM OLYMPIAD(SENIOR) 2019

Full Marks :100 **Time: $2\frac{1}{2}$ Hours**

Answer all questions. All questions carry equal marks.

1. Let a and b be positive real numbers and $a\sqrt{a}+b\sqrt{b}=183,\ a\sqrt{b}+b\sqrt{a}=182.$ Find $\frac{9}{5}(a+b)$.

Ekkufy;k a rFkk b okLrfod /kukRed la[;k;sa gSa] rFkk $a\sqrt{a}+b\sqrt{b}=183, a\sqrt{b}+b\sqrt{a}=182.$ rks $\frac{9}{5}(a+b)$ dk eku fudkysaA

Solution. Let $a=p^2$ and $b=q^2$ so that the given equations

Now $(p+q)^3=p^3+q^3+3pq(p+q)=183+546=729$

$\Rightarrow p+q=9.$

$\Rightarrow pq(p+q)=182$

$\Rightarrow pq=\frac{182}{9}$

Now $(a+b)=p^2+q^2=(p+q)^2-2pq=81-\frac{364}{9}$

$=\frac{729-364}{9}=\frac{365}{9}$

$\Rightarrow \frac{9}{5}(a+b)=\frac{9}{5}\times\frac{365}{9}=73$

2. If $2^x=3^y=6^{-z}$, then find the value of $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}$.

;fn $2^x=3^y=6^{-z}$, rks $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}$ dk eku fudkysaA

- **Solution:** If $2^x=3^y=6^{-z}$ then value of $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}$.

Let $2^x=3^y=6^{-z}=k$(say)

$\therefore 2=k^{\frac{1}{x}}, 3=k^{\frac{1}{y}}$ and $6=k^{-\frac{1}{z}}$

$\therefore$ here $2\times 3=6$

$\therefore k^{\frac{1}{x}}\times k^{\frac{1}{y}}=k^{-\frac{1}{z}} \quad \Rightarrow k^{\frac{1}{x}+\frac{1}{y}}=k^{-\frac{1}{z}}$

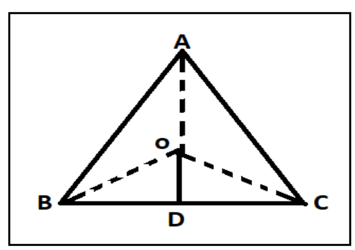
$\Rightarrow \frac{1}{x}+\frac{1}{y}=-\frac{1}{z}$

So, $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=0$

3. Let O be the incentre of a triangle ABC and D be a point on the side BC such that OD⊥ BC. If $\angle BOD=15^0$, then find $\angle ABC$.

Ekkufy;k O f=Hkqt ABC dk vUr%dsUæ gS rFkk Hkqtk BC ij D ,d ,slk foUnq gS fd OD⊥ BC. ;fn $\angle BOD=15^0$,rks $\angle ABC$ eku fudkysaA

Solution: we know that,



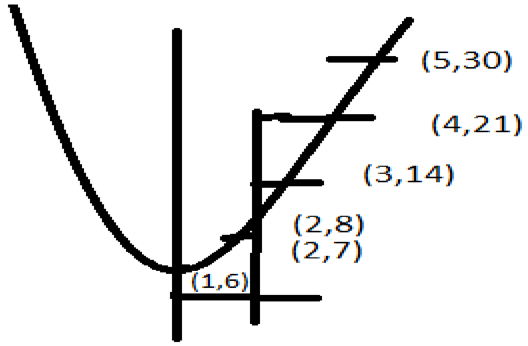
O is the any bisector of all three sides, in incentre. $OD\perp BC$

$\therefore \angle BDO=90^0$ and $\angle BOD=15^0$ (given)

$\therefore \angle DBO=180^0-(90^0+15^0)=75^0$

$\therefore \angle ABC=2\times\angle DBO=75^0\times 2=150^0$

4. **What is the sum of the squares of the roots of the equation $x^2 - 7[x] + 5 = 0$, where [x] is the maximum integer less than x?**
;fn $[x]$ vf/kdre iw.kkZad gks tks x ls de gS rks lehdj.k $x^2 - 7[x] + 5 = 0$ ds ewyksa d oxksZa ds ;ksx dk eku D;k gksxk ?
Solution:



$0 \le x < 1 \quad y_1 = x^2 + 5$

$y_2 = 7[x] - 0$

$1 \le x < 2 \quad y_1 = x^2 + 5$

$y_2 = 7$

$2 \le x < 3 \quad y_1 = x^2 + 5$

$y_2 = 14$

$3 \le x < 4 \quad y_1 = x^2 + 5$

$x = \sqrt{2}, x = \sqrt{23}$

Second Method:

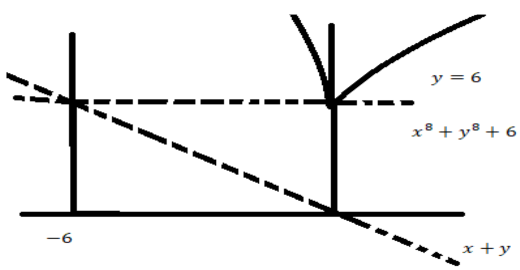
Solution: $x^2 - 7[x] + 5 = 0$
Any solution must satisfy the quadratic equation
$x^2 - 7x + 5 <= 0$
So x is between 0.8 and 6.2.
It follows that the possible values of $[x]$ are 1 to 6.
Let's go thru them one by one.
$[x]$ = 1, x^2 = 7 - 5 = 2 ---> x = sqrt(2) is a solution.
$[x]$ = 2, x^2 = 14 - 5 = 9 ---> x = 3 is not a solution because x has to be less than 3.
[x] = 3, x^2 = 21 -5 = 16 ---> x = 4 is not a solution
$[x]$ = 4, x^2 = 28 - 5 = 23 ---> x = sqrt(23) is a solution
$[x]$ = 5, x^2 = 35 - 5 = 30 ---> x = sqrt(30) is a solution
$[x]$ = 6, x^2 = 42 - 5 = 37 ---> x = sqrt(37) is a solution
There are altogether 4 distinct solutions.
They are sqrt(2), sqrt(23), sqrt(30), sqrt(37)
The sum of their squares is 2 + 23 + 30 + 37 = 32+60 = 92

Third Method:

$x^2 - 7[x] + 5 = 0 \Rightarrow [x] = \frac{x^2+5}{7} \Rightarrow \frac{x^2+5}{7} = k, k \in N \; Since \; 0 < x < 7$
$\because \; x^2 + 5 = 7k$
$\therefore x = \sqrt{2}, \sqrt{23}, \sqrt{30}, \sqrt{37}$
$\therefore$ Sum of squares of the roots of the equation $x^2 - 7[x] + 5 = 0$ is
$2 + 23 + 30 + 37 = 92$

5. **In a class the numbers of boys and girls are in the ratio 4 : 3. When 8 boys and 14 girls are absent, the number of boys is the square of the number of girls. Find the total number of students in the class.**
,d d{kk esa yM+ds ,oa yM+fd;ksa dh la[;kvksa dk vuqikr 4 % 3 gSA ;fn 8 yM+ds rFkk 14 yM+fd;k¡ vuqiLFkfr gksa rks yM+dksa dh la[;k yM+fd;ksa dh la[;k dk oxZ gSA d{kk esa dqy fo|kfFkZ;ksa dh la[;k Kkr djsaA
Solution. Let number of boys= $4n$ and number of girls= $3n$
Accordingly
$4n-8=(3n-14)^2=9n^2-84n+196$
$\Rightarrow 9n^2-84n-4n+196+8=0$
$\Rightarrow 9n^2-88n+204=0$
$\Rightarrow n=\frac{88\pm\sqrt{7744-7344}}{2\times 9}=\frac{88\pm 20}{2\times 9}=\frac{108}{18},\frac{68}{18}$
$\Rightarrow n=6,\frac{34}{9}$
Number of students in the class= $4n+3n=24+18=42$

6. **Find all real numbers satisfying $x^8+y^8=x+y-6$.**
$x^8+y^8=x+y-6$ dks larq'V dkjus okys lHkh okLrfod la[;kvksa dks Kkr djsaA
$\boldsymbol{Solution}$: $x^8+y^8=x+y-6$
$x^8+y^8+6=x+y$
$\because x^8+y^8+6=x+y$



If at all answer solution. It must be in II quadrant and for $x\le -6$
$\because x^8+y^8+6\ge 6$
But $y=-x$
$\because x^8+y^8+6$ given
$2(x^8)>2(-6)^8+6$
$\Rightarrow y=-x$ $\therefore$ No Real solution exist.

7. **How many six-digit numbers divisible by 25 can be formed using digits 0, 1, 2, 3, 4, 5 without repetition?**
0, 1, 2, 3, 4, 5 dks ysdj fcuk iqjko`fr ds N% vadksa dh fdruh la[;k cuk;h tk ldrh gS tks 25 ls foHkkT; gks
Solution: Digit to utilised without repetition$-$0,1,2,3,4,5
Now for divisibility by 25 last digit should be divisible by25
Last two digit must be 25 or 50

Case-I Last two digit is 25 then the remaining four digit can be selected in
$3 \times 3 \times 2 \times 1 = 18$
Case-II
Last two digit is 50 then the remaining four digit can be selected in
$4 \times 3 \times 2 \times 1 = 24$
Hence total number of six digit divisible by 25 $= 24 + 18$
$= 42$

8. Fin the remainder when 19^{92} is divided by 92.
;fn 19^{92} dk 92 ls Hkkx fn;k tk; rks “ks’k dk eku Kkr djsaA

Solution. Chinese Remainder theorem (along with other results). First note 92= 4 × 23 with gcd (4,23) =1. Let us call N= 19^{92}. We will compute, N(mod 4) and N (mod 23) and then use CRT to compute N (mod 92).

First, N (mod 4) = $(19)^{92}(\,mod\ 4) = (-1)^{92}(mod\ 4) = 1$

and $N(mod\ 23) = 19^4.[(19)^{22}\ (mod\ 23)]^2$

$(mod\ 23) = (-4)^4(mod\ 23) = (16)^4$

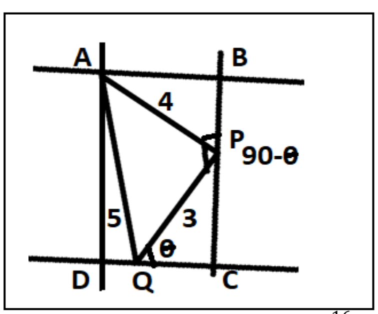
$(mod\ 23) = (-7)^2(mod\ 23) = 49$

$(mod\ 23) = 3.$

Note in the above we have used Fermat’s Little Theorem. Now, If you know CRT, you can directly say $N(\,mod\ 92) \ = \ 49.$

If not, you can compute it. One way to do it is write down two lists of numbers (one for each relation) and pick out the first common number.

9. Let $ABCD$ be a square. P and Q are any two points on BC and CD respectively such that $AP = 4\ cm$, $PQ = 3\ cm$, and $AQ = 5\ cm$. Find the side of the square.
Ekkufy;k $ABCD$,d oxZ gSA P rFkk Q Øe”k% BC rFkk CD ij nks ,sls foUnq gS fd $AP = 4\ cm$, $PQ = 3\ cm$ rFkk $AQ = 5\ cm$] oxZ dh Hkqtk dk eku fudkysaA



$AB = 4sin\theta$
$PB = 4cos\theta$
$PC = 3sin\theta$
$AB = PB \perp PC$
$4sin\theta = 4cos\theta + 3sin\theta$
$sin\theta = 4cos\theta \Rightarrow\ tan\theta = 4$
$tan\theta = 4 \Rightarrow sin\theta = \frac{4}{\sqrt{17}}$

$\therefore AB$ is side= $4sin\theta = \frac{16}{\sqrt{17}}$

10. If $(a^2-b^2)sin\theta+2ab\ cos\theta=a^2+b^2$, then find $tan\theta$.
;fn $(a^2-b^2)sin\theta+2ab\ cos\theta=a^2+b^2$, rks $tan\theta$ dk eku fudkysaA

Solution: The coefficient of $sin\theta = perpendicular$(p)

The coefficient of $cos\theta = base$(b)

and constant is *hypotenuse*= h

$\therefore\ \ tan\theta=\frac{p}{b}$

$\Rightarrow\ \ tan\theta=\frac{a^2-b^2}{2ab}$

TSTM Examination (Junior) Solution 2019

Full Marks :100 **Time: $2\frac{1}{2}$ Hours**

Answer all questions. All questions carry equal marks.

1. If $\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\left(1-\frac{1}{5^2}\right)\ \ldots\ldots\ldots.\left(1-\frac{1}{2019^2}\right)=\frac{x}{2019}$, then find the value of x.
;fn $\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\left(1-\frac{1}{5^2}\right)\ \ldots\ldots\ldots.\left(1-\frac{1}{2019^2}\right)=\frac{x}{2019}$, rks x dk eku Kkr djsaA
Solution: We have $\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\ldots\ldots..\left(1-\frac{1}{2019^2}\right)=\frac{x}{2019}$

$$\Rightarrow \frac{1}{2}\cdot\frac{3}{2}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{3}{4}\cdot\frac{5}{4}\ldots\ldots..\cdot\frac{2018}{2019}\cdot\frac{2020}{2019}=\frac{x}{2019}$$

$$\Rightarrow \frac{1}{2}\cdot\frac{2020}{2019}=\frac{x}{2019}$$

$$\Rightarrow x=1010$$

2. If $a^x=(x+y+z)^y,\ a^y=(x+y+z)^z\ and\ a^z=(x+y+z)^x$ then find the value of $x+y+z$, given that $a\neq 0$.
;fn $a^x=(x+y+z)^y,\ a^y=(x+y+z)^z$ rFkk $a^z=(x+y+z)^x$ rks $x+y+z$, dk eku Kkr djsa] tcfd $a\neq 0$.

Solution: $a^x.a^y.a^z=(x+y+z)^{x+y+z}$

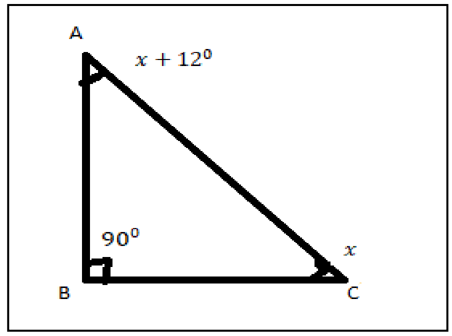
$a^{x+y+x}=(x+y+z)^{x+y+z}$

$x+y+z=a$

3. In a right angled triangle, the difference between two acute angle is $(\frac{\pi}{15})^c$. Find the angles in degree.

fdlh ledks.k f=Hkqt esa nksuksa U;wu dks.kksa dk vUrj $(\frac{\pi}{15})^c$ gSA dks.kksa dk eku fMxzh esa Kkr djsaA

Solution: $\therefore \angle ACB - \angle BAC = (\frac{\pi}{15})^C$



$= \frac{\pi}{15} \times \frac{180^0}{\pi} = 12^0$

Let , $\angle ACB = x, \therefore \angle BAC = x + 12^0$

$\therefore \boldsymbol{x} + \boldsymbol{x} + 12^0 = 90^0$

$\Rightarrow 2x = 78^0$

$\therefore x + 12^0 = 39^0 + 12^0 = 51^0$

Hence $\angle DCB = 39^0$ and $\angle BAC = 51^0$

4. Find natural numbers x, y such that $\sqrt{x} + y = 7$ and $x + \sqrt{y} = 11$.

   nks ç—frd la[;k x, y Kkr djsa tcfd $\sqrt{x} + y = 7$ rFkk $x + \sqrt{y} = 11$.

Solution:- $\sqrt{x} + y = 7$ ……………..(1)

and $x + \sqrt{y} = 11$ ………….(2)

Let $x = a^2$ and $y = b^2$ then from equation (1) is $a + b^2 = 7$ …………….(3)

and equation (2) is $a^2 + b = 11$ ……………….(4)

from equation (3)

$Put\ a = 7 - b^2$

$(7 - b^2)^2 + b = 11$

$b^4 - 14b^2 + 49 + b = 11$

$b^4 - 4b^2 - 10b^2 + 20b - 19b + 38 = 0$

$b^2(b^2 - 4) - 10b(b - 2) - 19(b - 2) = 0$

$(b - 2)\{b^3 + 2b^2 - 10b - 19\} = 0$

$b = 2$

Put value of $b = 2$ in (3) , we get

$a + (2)^2 = 7$

So $a = 3$

Therefore $x = a^2 = 9$ and $y = b^2 = 4$

5. If a, b, c are positive numbers such that $abc = 1$, then prove that $a + b + c \geq 3$ and $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq 3$.

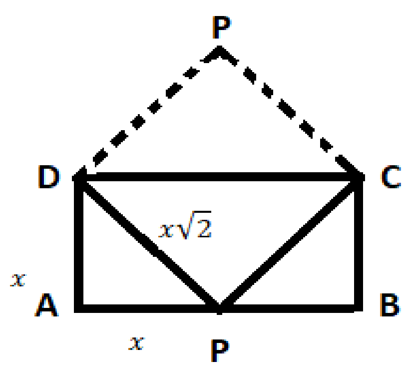
   ;fn a, b, c /kukRed la[;k;sa gSa tcfd $abc = 1$, rks lkfcr djsa fd $a + b + c \geq 3$ rFkk $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq 3$.

Solution : We know that $\frac{a+b+c}{3} \geq \sqrt[3]{\sqrt{abc}} \geq \frac{3}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}$

$\Rightarrow a + b + c \geq 3$ and $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq 3$

6. The length of a rectangular sheet of paper is twice its breadth. Show how to cut this paper into three pieces which can be rearranged to form a square.

   dkxt ds ,d vk;rkdkj i`'V dh yEckbZ mldh pkSM+bZ dh nqxquh gS bls ,sls rhu VqdM+ksa esa dSls dkVsa fd mUgsa ,d oxZ ds vkdkj esa j[kk tk ldsA



By cutting a ΔAPD and ΔBCP

P is mid-point of AB

7. If $a^2 + 1 = a$, then find the value of $a^{12} + a^6 + 1$.

   ;fn $a^2 + 1 = a$ gks rks $a^{12} + a^6 + 1$ dk eku Kkr djsaA

Given that

$a^2 + 1 = a \qquad \Rightarrow a + \frac{1}{a} = 1$ ……………(1)

Now, $a^{12} + a^6 + 1 = a^6(a^6 + 1 + \frac{1}{a^6})$

$= a^6(a^6 + \frac{1}{a^6} + 1)$ …………………(2)

Now, $= a^6 + \frac{1}{a^6} = = (a^3)^2 + \frac{1}{(a^3)^2}$

$(a^3 + \frac{1}{a^3})^2 - 2 = \{(a + \frac{1}{a})^3 - 3 \times 1 \times 1\}^2 - 2$

$\{ (1)^3 - 3\}^2 - 2 = 4 - 2 = 2$

$\therefore a^6(2+1) = 3a^6$

Hence, $a^{12} + a^6 + 1 = 3a^6$

We put, $a = 1$ it is satisfies the equation

$a^{12} + a^6 + 1 = 3$.

8. If $x = asec\theta cos\varphi$, $y = bsec\theta sin\varphi$, $z = ctan\theta$ then find the value of $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2}$.

   ;fn $x = asec\theta cos\varphi$, $y = bsec\theta sin\varphi$, $z = ctan\theta$ rks $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2}$ dk eku Kkr djsaA

   Solution: Given that

   $x = asec\theta.cos\varphi$,$y = bsec\theta.sin\varphi$ and $z = ctan\theta$

   $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c} = ?$

   $\therefore \frac{x^2}{a^2} = sec^2\theta.cos^2\varphi$, $\frac{y^2}{b^2} = sec^2\theta.sin^2\varphi$ and $\frac{z^2}{c^2} = tan^2\theta$

   $\therefore \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c} = sec^2\theta(cos^2\varphi + sin^2\varphi) - tan^2\theta$

   $= sec^2\theta - tan^2\theta = 1$ $\quad$ $[\because (cos^2\varphi + sin^2\varphi) = 1]$

9. Find the digit at the unit place in the number $2019^{2020} + 2020^{2019}$.
   la[;k $2019^{2020} + 2020^{2019}$ ds bdkbZ LFkku dk vad Kkr djsaA

   Solution: $9 = 9$

   $9 \times 9 = 81$

   $9 \times 9 \times 9 = 729$

   $9 \times 9 \times 9 \times 9 = 6561$

   Hence the value of unit place of $2019^{2020} = 1$

   and the value of unit place of $2020^{2019} = 0$

   $2019^{2020} + 2020^{2019} \rightarrow$ unit digit $= 1 + 0 = 1$

10. A man has 5 friends. In how many ways can he invite one or more of them in a party?

,d O;fDr dks 5 fe= gSaA fdrus çdkj ls og muesa ls ,d ;k vf/kd dks ,d ikVhZ esa vkeaf=r dj ldrk gS\

Solution: $5_{C_1} + 5_{C_2} + 5_{C_3} + 5_{C_4} + 5_{C_5}$

$$= 5 + \frac{5\times4}{1\times2} + \frac{5\times4}{1\times2} + 5 + 1$$

$$= 5 + 10 + 10 + 5 + 1$$

$$= 31$$